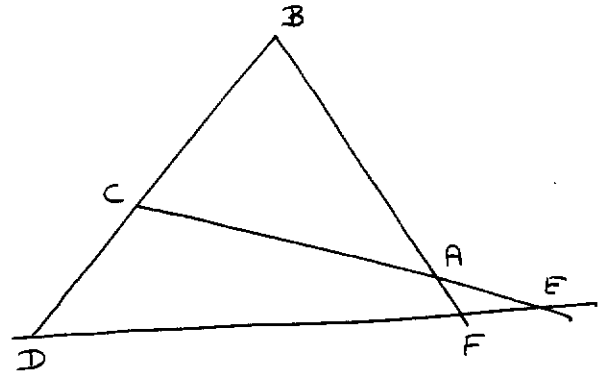
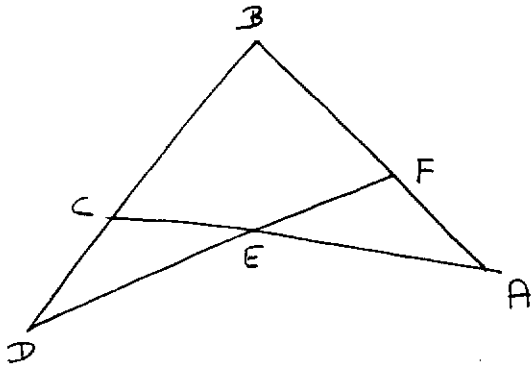


## Proving the theorem of Menelaos



For non-degenerate triangle  $ABC$  and  $D, E, F$  collinear as in above figures the theorem of Menelaos states

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$$

In terms of  $\lambda, \mu, r$ , defined by

$$D = \lambda B + (1-\lambda)C, \quad E = \mu C + (1-\mu)A, \quad F = rA + (1-r)B$$

we have to conclude

$$\frac{1-\lambda}{\lambda} \cdot \frac{1-\mu}{\mu} \cdot \frac{1-r}{r} = -1 \quad \text{or}$$

$$(0) \quad (1-\lambda) \cdot (1-\mu) \cdot (1-r) + \lambda \cdot \mu \cdot r = 0$$

from the fact that  $D, E,$  and  $F$  are collinear. This is expressed by

$$(1) \quad \text{Det. } P = 0 \quad \text{where} \quad P = \begin{vmatrix} D_x & D_y & 1 \\ E_x & E_y & 1 \\ F_x & F_y & 1 \end{vmatrix}$$

Writing  $D_x = \lambda B_x + (1-\lambda)C_x$ , etc. we can factorize

$$P = Q \cdot R \quad \text{where}$$

$$Q = \begin{vmatrix} \lambda & 1-\lambda & 0 \\ 0 & \mu & 1-\mu \\ 1-r & 0 & r \end{vmatrix} \quad R = \begin{vmatrix} B_x & B_y & 1 \\ C_x & C_y & 1 \\ A_x & A_y & 1 \end{vmatrix}$$

From this factorization we conclude

$$(2) \quad \text{Det. } P = (\text{Det. } Q) \cdot (\text{Det. } R)$$

Triangle ABC being non-degenerate is expressed by

$$(3) \quad \text{Det. } R \neq 0$$

and from (1), (2), (3) we conclude

$$\text{Det. } Q = 0$$

which, in view of Q's definition, equivaless (0).

I designed the above proof in reaction to a very classical argument in which the above two figures had to be dealt with separately. I like the proof for the way R enters the picture. (And just in case you feel tempted to send me shorter proofs of Menelaos's theorem: I know all about barycentric coordinates.)

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