# On two equations that have the same extreme solution

In [DS90], we have shown that for monotonic $f$ and sufficiently conjunctive $g$, equation

$$x: [f.x \Rightarrow g.x]$$

has a unique strongest solution. Also, if $f.x.y$ is monotonic in $x$ and in $y$, $f.x.x$ is monotonic in $x$. (In the following theorem, however, the existence of the strongest solutions is part of the premiss.)

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<u>Theorem</u> Let the strongest solution of $x: [f.x.x \Rightarrow g.x]$ (exist and) be $t$ ; let the strongest solution of $x: [f.x.t \Rightarrow g.x]$ (exist and) be $s$ ; then $[t \equiv s]$ if $f.x.y$ is monotonic in $y$.

<u>Proof</u> We are given

(0) $[f.t.t \Rightarrow g.t]$
(1) $[f.x.x \Rightarrow g.x] \Rightarrow [t \Rightarrow x]$ for all $x$
(2) $[f.s.t \Rightarrow g.s]$
(3) $[f.x.t \Rightarrow g.x] \Rightarrow [s \Rightarrow x]$ for all $x$
(4) $[f.x.p \Rightarrow f.x.q] \Leftarrow [p \Rightarrow q]$ for all $p, q, x$.

We now observe

† $[t \Rightarrow s]$
$\Leftarrow$ { (1) with $x := s$ }

$$
\begin{array}{ll}
 & [f.s.s \Rightarrow g.s] \\
\Leftarrow & \quad \{(2) \text{ and monotonicities}\} \\
 & [f.s.s \Rightarrow f.s.t] \\
\Leftarrow & \quad \{(4) \text{ with } p, q, x := s, t, s\} \\
\dagger & [s \Rightarrow t] \\
\Leftarrow & \quad \{(3) \text{ with } x := t\} \\
 & [f.t.t \Rightarrow g.t] \\
= & \quad \{(0)\} \\
 & \text{true}
\end{array}
$$

which observation proves, in view of the two lines marked $\dagger$, $[t \equiv s]$ by mutual implication.

(End of Proof.)

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The proof is very short, as befits such a small theorem. In fact, the theorem is so small that one may wonder why this EWD has been written at all. Here are my reasons.

- I did not really know the theorem.
- I did not know the proof, which with 1 step per given is a very nice example of a shortest possible proof.
- It makes clear that this theorem has nothing to do with fixpoints.
- It is always nice to start the new year with a theorem.

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