

Equilateral triangles and rectangular grids

The other day, I encountered the following problem: "Does there exist an equilateral triangle whose vertices have integer (orthogonal Cartesian) coordinates?". The problem is, indeed, as elementary as it looks, but in solving it, I had a few pleasant surprises; hence this note. I urge the reader to think about the problem before reading on.

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My first concern was how to characterize the notion "equilateral triangle": all edges of the same length, all angles of the same size, or a mixture of the two. Considering that, in analytical geometry, lengths are more readily expressed than angles, I chose the first characterization, i.e. upon the triangle with vertices $(0,0)$, (a,b) , (c,d) I imposed the requirement that its sides be of equal length. More precisely, my interest turned to integer quintuples (a,b,c,d,k) satisfying (0) \wedge (1) \wedge (2) with

$$(0) \quad a^2 + b^2 = k$$

$$(1) \quad c^2 + d^2 = k$$

$$(2) \quad (a-c)^2 + (b-d)^2 = k \quad .$$

Using (0) and (1), (2) can be simplified

to

$$(2') \quad 2 \cdot (a \cdot c + b \cdot d) = k$$

Seeing that k is even, we turn our attention in view of (0) and (1) to squares and factors of 2. And here we have to make a little jump. It does not suffice to observe (the correct) $\text{even} \cdot x^2 \equiv \text{even} \cdot x$, which for a sum of 2 squares only implies $\text{even} \cdot (x^2 + y^2) \equiv \text{even} \cdot x \equiv \text{even} \cdot y$. We have to reduce squares modulo 4 and use

$$(3a) \quad x^2 \pmod{4} = 0 \equiv \text{even} \cdot x$$

$$(3b) \quad x^2 \pmod{4} = 1 \equiv \text{odd} \cdot x$$

from which we conclude about the sum of squares

$$(4a) \quad (x^2 + y^2) \pmod{4} = 0 \equiv \text{even} \cdot x \wedge \text{even} \cdot y$$

$$(4b) \quad (x^2 + y^2) \pmod{4} = 1 \equiv \text{even} \cdot x \neq \text{even} \cdot y$$

$$(4c) \quad (x^2 + y^2) \pmod{4} = 2 \equiv \text{odd} \cdot x \wedge \text{odd} \cdot y.$$

And now we observe for an integer solution (a, b, c, d, k) of $(0) \wedge (1) \wedge (2')$:

$$\begin{aligned} & k \pmod{4} \neq 0 \\ = & \{ (2'), \text{ in particular even } k \} \\ & k \pmod{4} = 2 \\ = & \{ (0) \text{ and } (1) \} \\ & (a^2 + b^2) \pmod{4} = 2 \quad \wedge \quad (c^2 + d^2) \pmod{4} = 2 \\ = & \{ (4c) \text{ twice} \} \\ & \text{odd} \cdot a \quad \wedge \quad \text{odd} \cdot b \quad \wedge \quad \text{odd} \cdot c \quad \wedge \quad \text{odd} \cdot d \end{aligned}$$

$$\begin{aligned} &\Rightarrow \{ \text{arithmetic} \} \\ &\text{even. } (a \cdot c + b \cdot d) \\ &= \{ (2^i) \} \\ &k \pmod{4} = 0 \end{aligned}$$

Having thus proved

$$(5) \quad k \pmod{4} = 0 ,$$

we conclude on account of (0), (1), (4a)

$$(6) \quad \text{even.} a \wedge \text{even.} b \wedge \text{even.} c \wedge \text{even.} d .$$

If (a, b, c, d, k) is an integer solution of $(0) \wedge (1) \wedge (2)$, $(a/2, b/2, c/2, d/2, k/4)$ obviously solves that equation as well; (5) and (6) tell us now that the latter is again an integer solution. By mathematical induction we conclude that any solution of $(0) \wedge (1) \wedge (2)$ is divisible by any power of 2 (4), and hence equal to $(0, 0, 0, 0, 0)$, 0 being the only integer with an unbounded number of divisors. All zeros is indeed a solution, i.e. there exist equilateral triangles whose vertices have integer coordinates, but such a triangle is degenerate and its vertices coincide.

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For several reasons I was pleased with the above argument. Of course I regret the rabbit

embodied by (3a)(3b), but I console myself with the thought that it is a fairly small one. The proof of $k \pmod{4} = 0$ is a nice example of proving \overline{P} by constructing a weakening chain from $\neg P$ to P , and the final part of the argument nicely exploits that only 0 has an unbounded number of divisors.

Remark The latter observation can be turned into a heuristic: a good way of showing that a Diophantine equation has no other solution than 0, it suffices to show that any solution p can be divided by some n ($n > 1$) such that p/n is again a solution. (End of Remark)

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