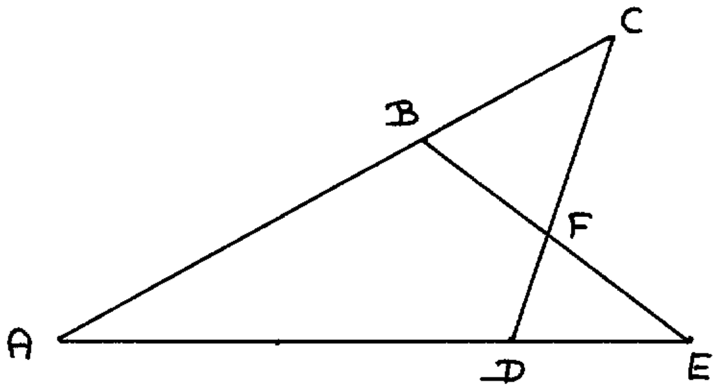
A Geometry Problem from "The Monthly", March 1998

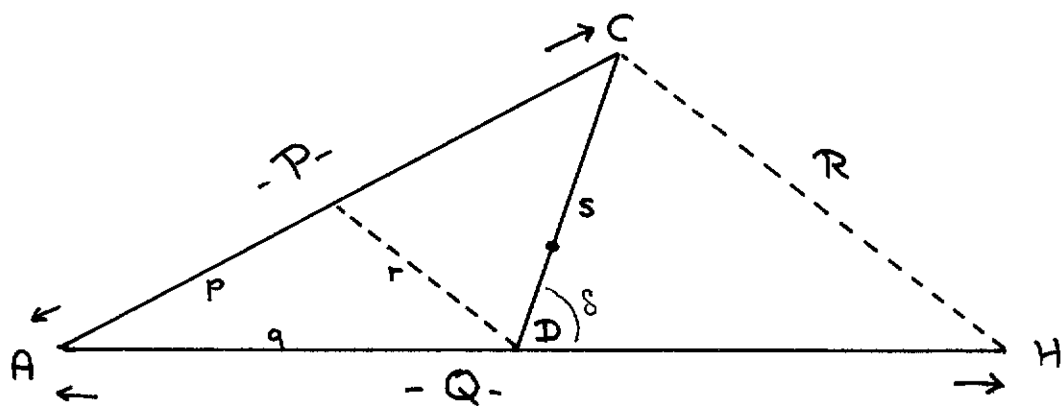
In a review in The American Mathematical Monthly, March 1998, the following problem was stated. Prove that in a figure like



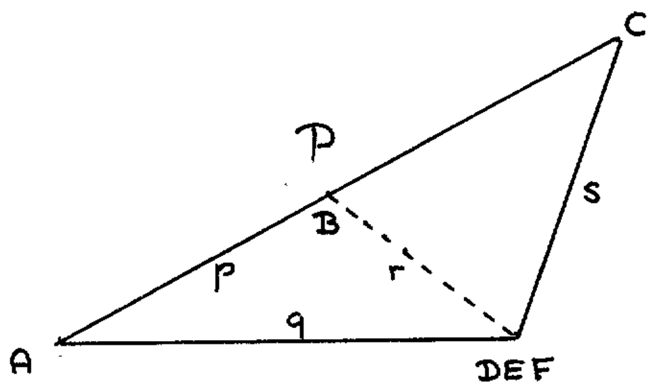
$$AB + BF = AD + DF \Rightarrow AC + CF = AE + EF \quad .$$

* * *

After a while I found myself considering the figures with the points A,C,D fixed but the line BFE moving parallel to itself, with the dotted lines through D and C its extreme positions:



Below we give the positions of B, F, E when the moving line is in its leftmost position — i.e. coinciding with the dotted line through D — and later when the moving line is in its rightmost position.



First we are interested in the values of α and γ when the moving line is in this leftmost position, and where α and γ — corresponding to antecedent and consequent respectively — are given by

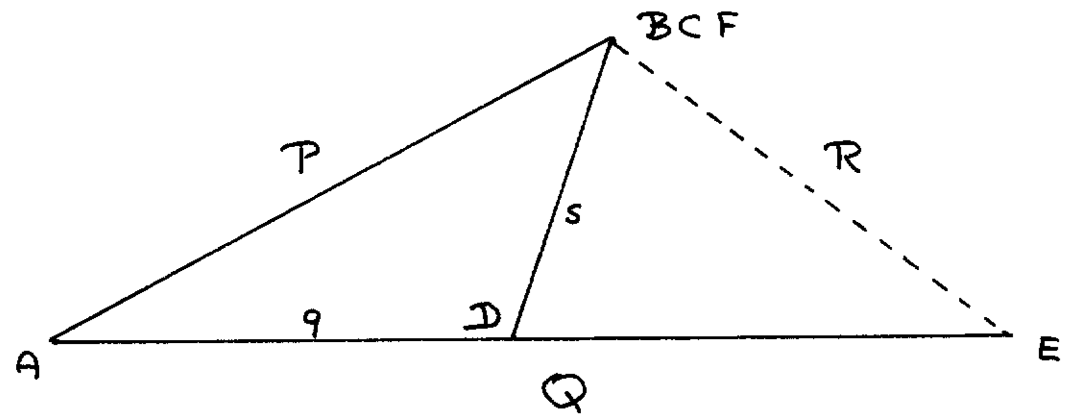
α = AB + BF − AD − DF and

γ = AC + CF − AE − EF .

Denoting these leftmost values by $\alpha 0$ and $\gamma 0$, we read from the above figure

(0) $\alpha 0 = p + r - q$, $\gamma 0 = P + s - q$.

Next we are interested in the values of α and γ, when the moving line is in its rightmost position. The points B, F, E are then located as shown in the following figure



Calling these values of α and γ, $\alpha 1$ and $\gamma 1$ respectively, we read from the above picture

(1) $\alpha 1 = P - q - s$, $\gamma 1 = P - Q - R$.

Our proof obligation is $\alpha = 0 \Rightarrow \gamma = 0$. Actually, and not surprisingly, we shall show $\alpha = 0 \equiv \gamma = 0$. We can do so because all components of α and γ, and hence α and γ themselves are linear functions of the position of the moving line (measured in distance from, say, the leftmost dotted line). Because $\alpha 0 > 0$ and $\alpha 1 < 0$, these two values determine by linear interpolation the unique position at which $\alpha = 0$, and similarly for γ. We don't need to determine these positions, we only need to show that they are the same, which follows from

$$\alpha 0 \cdot \gamma 1 = \alpha 1 \cdot \gamma 0 \quad .$$

Thanks to (0) and (1), the latter proof obligation is equivalent to

$$(p-q+r)\cdot(P-Q-R) = (P-q-s)\cdot(P-q+s)$$

Because (in the 2nd figure) the dotted lines are parallel, we have

(2) $p\cdot Q = q\cdot P \qquad q\cdot R = r\cdot Q \qquad r\cdot P = p\cdot R$

and the above proof obligation can be simplified to

(3) $p\cdot P + q\cdot Q - r\cdot R = P^2 + q^2 - s^2$.

We are getting on more and more familiar ground. Using the Cosine Rule – i.e. "$c^2 = a^2 + b^2 - 2\cdot a\cdot b\cdot \cos.\gamma$" – we deduce in the 2nd figure

true

$\equiv$ { Cosine Rule in Δ ACD and Δ CHD }

$P^2 = q^2 + s^2 + 2\cdot q\cdot s\cdot \cos.\delta \quad \wedge$

$R^2 = (Q-q)^2 + s^2 - 2\cdot(Q-q)\cdot s\cdot \cos.\delta$

$\Rightarrow$ { eliminate $\cos.\delta$ }

$$(Q-q)\cdot P^2 + q\cdot R^2 = q\cdot Q\cdot(Q-q) + Q.s^2$$
$\equiv$ { distribution and rearranging }
$$Q\cdot P^2 + Q\cdot q^2 - Q\cdot s^2 = q\cdot P^2 + q\cdot Q^2 - q\cdot R^2$$
$\equiv$ { (2) and $Q \neq 0$ }
$$P^2 + q^2 - s^2 = p\cdot P + q\cdot Q - r\cdot R$$
$\equiv$ { def. of (3) }
(3)

and this completes my proof.

Note Introduction and elimination of cos. δ as in the last calculation is a standard device. (End of Note.)

The recognition of the linear dependence on the displacement of the moving line was crucial. The separate naming of p, q, r and of P, Q, R freed us from mentioning their constant ratio where it did not matter.

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